Problem: If A, B & C are sets such that $A \cup B = A \cup C$ & $A \cap B = A \cap C$, show that $B = C$.

Solution:

Let us consider an arbitrary element $b \in B$.

There only 2 possibilities w.r.t this element – either $b \in A$ as well or $b \notin A$.

Let us consider these 2 cases separately:

Case 1: $b \in B$ & $b \in A$

$\therefore b \in A \cap B$ [By definition]

$\therefore b \in A \cap C$ [Since $A \cap B = A \cap C$]

$\therefore b \in A$ & $b \in C$ [By definition]

$\therefore b \in C$

Case 2: $b \in B$ & $b \notin A$

$\therefore b \in A \cup B$ [By definition]

$\therefore b \in A \cup C$ [Since $A \cup B = A \cup C$]

$\therefore b \in A$ or $b \in C$ or both [By definition]

But $b \notin A$

$\therefore b \in C$

In both cases, $b \in C$ [for an arbitrary $b \in B$]

$\therefore$ $B \subset C$ [By definition]

|||^{ly}, by choosing an arbitrary element $c \in C$ and applying a similar logic, we can show that $C \subset B$

$\therefore$ $\underline{\underline{B = C}}$