Problem: Let A & B be two sets. If $A\cap X = B\cap X = \phi$ & $A\cup X = B\cup X$ for a given set X, show that $A = B$.

Solution:

To show that $A = B$, we need to show that $A \subset B$ & $B \subset A$.

To show that $A \subset B$, we need to take an arbitrary element of A & show that it belongs to B as well.

Let m be an arbitrary element of A

i.e, $m \in A$

Then, by definition of the union operator

$m \in A\cup X$

& since $A\cup X = B\cup X$ [Given]

$m \in B\cup X$ as well.

$\therefore m \in B$ or $m \in X$ or both — (1)

But since $m \in A$ (assumed) & $A\cap X = \phi$ (Given), by definition of the intersection operator $m \notin X$ $[\because$ if $m \in A$ & $m \in X$, $A \cap X \neq \emptyset]$

$\therefore$ From (1), since $m \notin X$,

$m \in B$

We have shown that an arbitrary element of A is an element of B as well.

$\therefore$ $A \subset B$

|||ly, we can show that $B \subset A$
[By making use of the fact that $B \cup X = A \cup X$ & $B \cap X = \emptyset$]

$\therefore$ $\underline{A = B}$