<u>Problem</u>: A function $f: \mathbb{R} - \{\frac{2}{3}\} \to \mathbb{R} - \{\frac{4}{3}\}$ is defined by $f(x) = \dfrac{4x+5}{3x-2}$

(a) Is this function bijective? Why / Why not?

(b) Find the inverse of the function if it exists.

<u>Solution</u>:

(a) For a function to be bijective, it must be both one-one & onto.

f is one-one if no two elements of its domain map onto the same element under f

Let x_1 & x_2 be two arbitrary elements $\in$ the domain of f.

Let us assume that they map onto the same element

i.e., $f(x_1) = f(x_2)$

$$\therefore \quad \frac{4x_1+5}{3x_1-2} = \frac{4x_2+5}{3x_2-2}$$

$$\Rightarrow \cancel{4x_1(3x_2)} - 8x_1 + 15x_2 - \cancel{10}$$
$$= \cancel{4x_2(3x_1)} - 8x_2 + 15x_1 - \cancel{10}$$

$$\Rightarrow 23x_1 = 23x_2$$

$$\Rightarrow x_1 = x_2$$

We have shown that two distinct elements of the domain can't map onto the same element

$\therefore$ f is one-one

f is onto if all elements in the co-domain of f have a pre-image in the domain under f.

Let $y \in$ co-domain $\mathbb{R} - \left\{\frac{4}{3}\right\}$ be an arbitrary element of the co-domain.

If we assume that there exists an element, say $x, \in$ domain $\mathbb{R} - \left\{\frac{2}{3}\right\}$ such that

$$f(x) = y$$

i.e., $\dfrac{4x+5}{3x-2} = y$

To verify our assumption about the existence of a x, we will try to express x in terms of y.

Doing this, we get

$$4x+5 = (3x-2)y = 3xy - 2y$$

$$\Rightarrow 3xy - 4x = 2y+5$$

$$\Rightarrow x(3y-4) = 2y+5$$

$$\Rightarrow x = \frac{2y+5}{3y-4} \quad -(1)$$

$\frac{2y+5}{3y-4}$ is defined except when the denominator is 0.

$\therefore \frac{2y+5}{3y-4}$ makes sense for all y except when $3y-4=0$

$\therefore \frac{2y+5}{3y-4}$ exists for all y except $y = \frac{4}{3}$

But co-domain of f is $R - \{\frac{4}{3}\}$

$\therefore$ For any $y \in$ the co-domain of f, there exits an element, say x, $\in$ the domain such that

$$f(x) = y$$

$\therefore$ f is onto.

We have shown that f is one-one & onto.

$\therefore$ f is bijective.

(b) Since f is bijective, its inverse exists.

From ①, given a $y \in$ the co-domain of f, we have a way of finding element x which maps onto it under f.

$\therefore$ If consider a function

$g : R - \{\frac{4}{3}\} \to R - \{\frac{2}{3}\}$ such that

$$g(y) = \frac{2y+5}{3y-4}$$

Then, applying f on an element, say x of the domain will give its image, say y and applying g on this should g us back the element x that we started with.

In other words, g is the inverse of f.

Let us establish this formally.

g will be the inverse of f if

$$g \circ f = I_X \quad \text{and} \quad f \circ g = I_Y$$

Let us see whether this is true.

Given $x \in R - \left\{\frac{2}{3}\right\}$ [domain of f]

$$g \circ f(x) = g(f(x)) = g\left(\frac{4x+5}{3x-2}\right)$$

$$= \frac{2\left(\frac{4x+5}{3x-2}\right)+5}{3\left(\frac{4x+5}{3x-2}\right)-4} = \frac{8x+\cancel{10}+15x-\cancel{10}}{\cancel{12x}+15-\cancel{12x}+8}$$

$$= \frac{23x}{23} = x$$

$\therefore g \circ f = I_X$

Let $y \in \mathbb{R} - \left\{\frac{4}{3}\right\}$ [Domain of g]

Then $f \circ g(y) = f(g(y)) = f\left(\frac{2y+5}{3y-4}\right)$

$$= \frac{4\left(\frac{2y+5}{3y-4}\right) + 5}{3\left(\frac{2y+5}{3y-4}\right) - 2} = \frac{8y + \cancel{20} + 15y - \cancel{20}}{\cancel{6y} + 15 - \cancel{6y} + 8}$$

$$= \frac{23y}{23} = y$$

$\therefore f \circ g = I_Y$

$\therefore$ g is the inverse of f.