Problem: $f: \mathbb{N} \rightarrow \mathbb{N}$ is a function defined by

$$f(x) = \frac{x+1}{2}, \text{ if } x \text{ is odd}$$

$$f(x) = \frac{x}{2}, \text{ if } x \text{ is even}$$

(a) Is the function one-one? Why / Why not?

(b) Is the function onto? Why / Why not?

Solution:

(a) f is one-one only when no two elements of its domain map onto the same element.

Consider 1 & $2 \in \mathbb{N}$

$$f(1) = \frac{1+1}{2} = 1 \quad [\because 1 \text{ is odd}]$$

$$f(2) = \frac{2}{2} = 1 \quad [\because 2 \text{ is even}]$$

$$\therefore f(1) = f(2) = 1$$

Since, we have shown that two elements of the domain map onto the same element, f is not <u>one-one</u>.

(b) f is onto only when all elements of its co-domain have a pre-image under f.

Let us consider an arbitrary $y \in$ the co-domain $\mathbb{N}$ of the function f.

To find out whether an element, say x, exists in the domain of f such that $f(x) = y$, we need to consider 2 cases.

Case 1 Could x be a odd natural number?

If x is odd $f(x) = y$

$$\Rightarrow \frac{x+1}{2} = y$$

$\Rightarrow x = 2y - 1$ which is an odd natural number for all values of $y \in \mathbb{N}$

$\therefore$ There exists an odd natural number x such that $f(x) = y$ for any $y \in \mathbb{N}$.

Case 2 : Could x be an even natural number?

If x is even,

$$f(x) = y$$

$$\Rightarrow \frac{x}{2} = y$$

$\Rightarrow x = 2y$ which is an even natural number for all values of $y \in \mathbb{N}$

$\therefore$ there exists an even natural number, say x, such that $f(x) = y$ for any $y \in \mathbb{N}$

Since we have shown that there exists a pre-image for all elements of the co-domain of f, f is <u>onto</u>.

In fact, for any element of the co-domain there exist 2 pre-images under f & this is what makes the function not one-one.